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Stochastic Strings, Topology, and Space-Time Confinement

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Stochastic space-time caused by random strings is considered. By using a conformlike transformation of the metric, we reconstruct "gravitational" theory and derive its consequences. Such an approach permits us to find natural quark confinement due to induced gravitation and to take into account the topological structure of space-time in any physical quantity.

Recently, much attention has been paid to the study of space-time structure at short distances. It has been assumed that at small distances space-time may take different structures, such as quantum or discrete (Wilson, 1974; Lee, 1983; Friedberg and Lee, 1983; Fradkin and Tseytlin, 1985; Yamamoto, 1985; Banai, 1984, 1985; Fujiwara, 1980), foamlike (Wheeler, 1964; Hawking, 1978, 1983; Strominger, 1984; Misner *et al.*, 1973), code (Finkelstein, 1969, 1972, 1974), cellular (Kirilov and Kochnev, 1987; Cole, 1972), and so on. Among them the stochastic or fluctuational character of space-time may become the most probable candidate and the natural arena of future physical theory (for example, Namsrai, 1986; Prugovecki, 1984; and references therein). Indeed, if one believes in the quantum principle and Einstein's theory, then stochastic or fluctuational properties of space-time should inevitably appear in the microworld.

Stochastic or quantum geometry plays an important role in representing gauge theories by random surfaces and strings (Polyakov, 1981; Gomez, 1982) and in the construction of a unified theory of elementary particle interactions based on the theory of strings and superstrings [see, for details, Green *et al.* (1987)].

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In this paper we present a simple scheme of introducing stochastic space-time induced by random strings. Let us consider a bosonic string, the motion of which can be described by a two-dimensional surface $\mathcal{M}$ known as the string world-sheet. The latter is embedded in a d-dimensional space-time (here we consider the case $d=4$). The manifold $\mathcal{M}$ is parametrized by coordinates

$$\sigma=\sigma^a \equiv \{\sigma^1=\sigma, \sigma^2=\tau\}$$

and is equipped with a metric tensor g_{ab} while space-time has coordinates $x^\mu(\tau)$ and metric $G_{\mu\nu}$.

Further, we propose that coordinates of strings $y^\mu(\sigma)$ are random variables, the behavior of which is described by a probability distribution

$$P[y]=\frac{1}{N}\exp\left\{-\frac{1}{2}\int_{\mathcal{M}_1}\int_{\mathcal{M}_2} d^2\sigma_1\, d^2\sigma_2 \sqrt{g_1}\sqrt{g_2}\, y^\mu(\sigma_1) D^{-1}_{\mu\nu}(\sigma_1-\sigma_2) y^\nu(\sigma_2)\right\} \tag{1}$$

where N is a constant chosen so that $P[y]$ is normalized to unity and $D^{-1}_{\mu\nu}$ is the inverse of the two-point correlation

$$\langle y^\mu(\sigma_1) y^\nu(\sigma_2)\rangle = D^{\mu\nu}(\sigma_1-\sigma_2) \tag{2}$$

Our main assumption is that due to the presence of the random string field, space-time begins to fluctuate and its topological structure gives rise to changes in physical quantities at short distances. In order to introduce stochastic fluctuations in the metric induced by the random strings, we define the conformlike transformation of coordinates leading to the passage from the usual local inertial system of reference ξ^α with the Minkowski metric $\eta_{\alpha\beta}$ to the quasilocal system of "averaged" (pointlike) string coordinates x^μ:

$$y^\mu(\sigma^1, \sigma^2)=(1/\sqrt{g}\, R)\delta(\sigma^1)x^\mu(\sigma^2)+\eta^\mu(\sigma^1, \sigma^2)$$

with an induced stochastic metric $G_{\mu\nu}(x, y)$. Here functions $\eta^\mu(\sigma^a)$ are some random variables of the same type of $y^\mu(\sigma^a)$. The variables ξ^α and x^m depend on the proper time τ. Let us consider the formal transformation

$$\xi^\alpha \Rightarrow \xi^\mu = x^\mu \exp\left\{\frac{1}{2(\pi\alpha')^{1/2}}\int_{\mathcal{M}} d^2\sigma \sqrt{g}\, R U_\mu(\sigma) y^\mu(\sigma)\right\} \tag{3}$$

where R is the Ricci curvature scalar of the manifold $\mathcal{M}$ and $U_\mu(\sigma)$ is a unit vector

$$-\eta^{\mu\nu} U_\mu(\tau) U_\nu(\tau) = 1 \tag{4}$$

depending on the timelike variable $\sigma^2 = \tau$ only. The differential of the string coordinates is given by

$$\frac{\partial y^\mu(\sigma)}{\partial x^\nu(\tau)} = \delta^\mu_\nu \frac{1}{\sqrt{g}\,R} \delta(\sigma^1)\delta(\sigma^2 - \tau)$$

Here the appearance of the value $R\sqrt{g}$ follows from a dimensional argument and the normalization condition of $\delta^2(\sigma)$, which is integrated to 1 with measure $d^2\sigma\sqrt{g}$.

Now the rectilinear trajectory of a particle given by the equation

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0, \qquad d\tau^2 = -\eta_{\alpha\beta}\, d\xi^\alpha\, d\xi^\beta \tag{5}$$

in the system of reference ξ^α takes the form

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \tag{6}$$

where $\Gamma^\lambda_{\mu\nu}$ is the affine connection-like quantity defined as

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu\, \partial x^\nu} \tag{7}$$

The proper time (5) may also be expressed in the system of reference x^μ with the stochastic metric $G_{\mu\nu}(x, y)$ by the formula

$$d\tau^2 = -\eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}\, dx^\mu\, dx^\nu \tag{8}$$

or

$$d\tau^2 = -G_{\mu\nu}(x, y)\, dx^\nu\, dx^\mu \tag{9}$$

where $G_{\mu\nu}(x, y)$ is the stochastic metric defined as

$$G_{\mu\nu}(x, y) = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \tag{10}$$

Making use of connection (3), we calculate the explicit form of (10) and verify its identify

$$\frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial x^\lambda}{\partial \xi^\alpha} = \delta_\mu^\lambda \qquad \text{or} \quad G_{\mu\nu} G^{\nu\lambda} = \delta_\mu^\lambda \tag{11}$$

Direct calculation gives

$$\frac{\partial \xi^\alpha}{\partial x^\mu} = \left[\delta_\mu^\alpha + \frac{1}{2} \varepsilon_\mu^\alpha \right] \cdot \Lambda \tag{12}$$

where

$$\Lambda = \exp\left\{ \frac{1}{2} \frac{1}{\sqrt{\pi \alpha'}} \int_{\mathcal{M}} d^2\sigma \sqrt{g}\, R U_\mu(\sigma) y^\mu(\sigma) \right\}$$

and

$$\varepsilon_\mu^\alpha(x) = \frac{1}{\sqrt{\pi\alpha'}} x^\alpha(\tau) U_\mu(\tau)$$

An inverse Jacobian of transformation with respect to (12) is

$$\frac{\partial x^\lambda}{\partial \xi^\alpha} = \Lambda^{-1}[\delta_\alpha^\lambda - \tfrac{1}{2}\varepsilon_\alpha^\lambda(x) + \tfrac{1}{4}\varepsilon_\alpha^\rho(x)\varepsilon_\rho^\lambda(x) - \tfrac{1}{8}\varepsilon_\alpha^\rho(x)\varepsilon_\rho^\delta(x)\varepsilon_\delta^\lambda(x) + \cdots] \tag{13}$$

Identities (12) and (13) allow us to define the metric tensor and its inverse by the formulas

$$G_{\mu\nu}(x, y) = \Lambda^2[\eta_{\mu\nu} + \varepsilon_{\mu\nu}(x) + \tfrac{1}{4}\varepsilon_\mu^\rho(x)\varepsilon_{\nu\rho}(x)] \tag{14a}$$

and

$$G^{\nu\sigma}(x, y) = \Lambda^{-2}[\eta^{\nu\sigma} - \varepsilon^{\nu\sigma}(x) + \tfrac{3}{4}\varepsilon^{\nu\rho}(x)\varepsilon_\rho^\sigma(x) - \cdots] \tag{14b}$$

It is easily verified that

$$G^{\nu\sigma}(x, y) G_{\beta\nu}(x, y) = \delta_\beta^\alpha$$

Next it is necessary to carry out an averaging procedure in (14a) and (14b) over random variables $y^\mu(\sigma)$ with the probability distribution (1). For the two-point correlation function (2) we use the white noise covariance

$$D^{\mu\nu}(\sigma_1 - \sigma_2) = -\eta^{\mu\nu} \frac{\lambda^2}{\sqrt{g}\, R} \delta^2(\sigma_1 - \sigma_2) \tag{15}$$

where $\eta_{\mu\nu}=\eta^{\mu\nu}$ is the Minkowski metric tensor defined as $\eta_{\mu\nu}=0$ for $\mu=\nu$ and $-\eta_{00}=\eta_{11}=\eta_{22}=\eta_{33}=1$, and λ is some constant dimension of length. Here we distinguish two possibilities:

(a) $\lambda^2\sim G^2$, where G is the Newtonian constant.
(b) $\lambda^2\sim(\alpha')^2$, where α' is the inverse string tension (a size of the string).

The former means that fluctuation of the string coordinates takes place at the Planck scale, while the second case means that coordinates obey random properties in a domain characterized by the size of the string.

Taking into account formulas (4) and (15) and using the Feynman rules

$$\left\langle \exp\int d^4x\, K(x)\psi(x)\right\rangle_{\psi}=\exp\left(\frac{1}{2}\int\int d^4x\, d^4y\, K(x)\Delta(x-y)K(y)\right) \quad (16)$$

we have

$$\langle G_{\mu\nu}(x,y)_y=[\eta_{\mu\nu}+\varepsilon_{\mu\nu}(x)+\tfrac{1}{4}\varepsilon^{\rho}_{\mu}(x)\varepsilon_{\nu\rho}(x)]\exp\left(\frac{\lambda^2}{2\pi\alpha'}\int_{\mathscr{M}} d^2\sigma\,\sqrt{g}\,R\right) \quad (17)$$

where

$$\mathrm{Euler}(\mathscr{M})=\frac{1}{4\pi}\int_{\mathscr{M}} d^2\sigma\,\sqrt{g}\,R \quad (18)$$

is a topological invariant known as the Euler characteristic (or the Euler number) of $\mathscr{M}$. If $\mathscr{M}$ has genus N (i.e., if $\mathscr{M}$ is homeomorphic to a sphere with N handles, or to a connected sum of N tori), then

$$\mathrm{Euler}(\mathscr{M})=2-2N$$

Thus, the physical space-time metric at large distances is obtained by a sum over equivalent topological structures,

$$G_{\mu\nu}(x)=\sum_N\langle G_{\mu\nu}(x,y)\rangle_y=[\eta_{\mu\nu}+\varepsilon_{\mu\nu}(x)+\tfrac{1}{4}\varepsilon^{\rho}_{\mu}(x)\varepsilon_{\rho\nu}(x)]\times I \quad (19)$$

where the multiplier factor

$$I=\frac{e^{4\Delta}}{1-e^{-4\Delta}},\qquad \Delta=\frac{\lambda^2}{\alpha'}$$

has appeared due to the topological structure of space-time at small distances.

Further, by using the general covariant method (Weinberg, 1972) of the description of gravitational phenomena in space-time with stochastic and quantum metrics in the weak-field limit (Namsrai, 1991) having the system of reference x^{μ} with metric (14a), we can easily construct the theory of induced "gravity" caused by random strings and to reanalyze its consequences. These results will be given elsewhere. Here we concentrate on two interesting facts.

First, we know (Namsrai, 1991; Landau and Lifschitz, 1971) that due to stochastic fluctuation of the space-time metric (14), in the limiting case when the velocity of the quark particle is small, an additional nonrelativistic "potential" φ_f also appears:

$$\varphi_f = \tfrac{1}{2}c^2(-1 - G_{00}) \tag{20}$$

As the unit vector $U_\mu(\tau)$ in (4) we choose the four-velocity of the particle (quark) and derive that

$$\langle \varphi_f \rangle = \frac{c^2}{2}\left[-1 + \frac{e^{4\Delta}}{1 - e^{-4\Delta}}\left(1 - \frac{\mathbf{x}^2}{4\pi\alpha'}\right)\right], \qquad \Delta = \frac{\lambda^2}{\alpha'}$$

in the stationary case. Thus, the induced force is

$$\mathbf{F}_f = -\nabla\varphi_f = \left(c^2 \frac{r}{4\pi\alpha'}\mathbf{n}\right)\frac{e^{4\Delta}}{1 - e^{-4\Delta}}, \qquad \mathbf{n} = \mathbf{r}/r \tag{21}$$

Second, when the quark particle moves in the constant fictitious "field" $\varepsilon_{\mu\nu}(\mathbf{x})$ in (14) its averaged energy is defined as

$$E_f = \frac{mc^2}{\sqrt{1 - v^2/c^2}}\sqrt{-G_{00}} = \frac{mc^2}{\sqrt{1 - v^2/c^2}}\frac{e^{\Delta}}{1 - e^{-\Delta}} \times \left(1 - \frac{\mathbf{x}^2}{4\pi\alpha'}\frac{1}{1 - v^2/c^2}\right)^{1/2}$$

From this we immediately conclude that the quark undergoes a finite motion, the phase diagram (Figure 1) of which is defined as

$$\frac{\mathbf{p}^2}{p_{\max}^2} + \frac{\mathbf{x}^2}{x_{\max}^2} \le 1 \tag{22}$$

where $p_{\max} = mc$ and $x_{\max} = 2\sqrt{\pi\alpha'}$. Assuming $\alpha' = m_p^{-2}$, we get $x_{\max} = 10^{-13}$ cm.

CONCLUSION

Thus, we observe that due to stochastic fluctuation of the space-time metric caused by random strings the quarks are exactly confined inside the domain characterized by the string tension parameter α', and the force between them obeys the linear law distance.

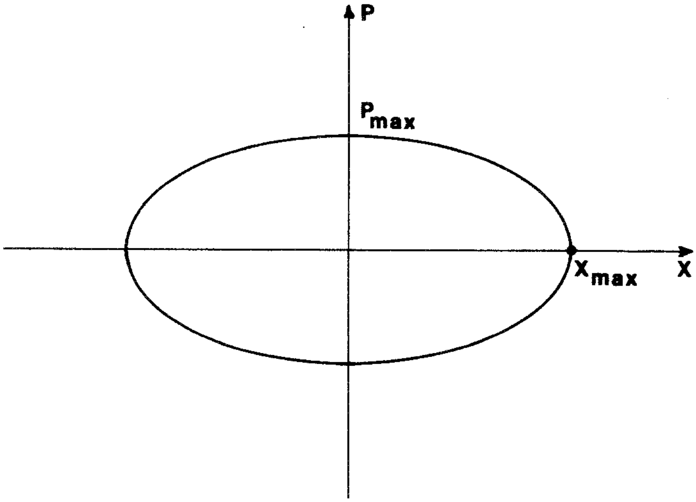

Fig. 1. Phase diagram for a quark moving in the stochastic space-time induced by random strings.

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